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Relativistic hydrodynamics from the single-generator bracket formalism of nonequilibrium thermodynamics

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Abstract: We employ the generalized bracket formalism of nonequilibrium thermodynamics by Beris and Edwards to derive Lorentz-covariant time-evolution equations for an imperfect fluid with viscosity, dilatational viscosity, and thermal conductivity. Following closely the analysis presented by Öttinger (Physica A, 259, 1998, 24–42; Physica A, 254, 1998, 433–450) to the same problem but for the GENERIC formalism, we include in the set of hydrodynamic variables a covariant vector playing the role of a generalized thermal force and a covariant tensor closely related to the velocity gradient tensor. In our work here, we derive first the nonrelativistic equations and then we proceed to obtain the relativistic ones by elevating the thermal variable to a four-vector, the mechanical force variable to a four-by-four tensor, and by representing the Hamiltonian of the system with the time component of the energy-momentum tensor. For the Poisson and dissipation brackets we assume the same general structure as in the nonrelativistic case, but with the phenomenological coefficients in the dissipation bracket describing friction to heat and viscous effects being properly constrained for the resulting dynamic equations to be manifest Lorentz-covariant. The final relativistic equations are identical to those derived by Öttinger but the present approach seems to be more general in the sense that one could think of alternative forms of the phenomenological coefficients describing friction that could ensure Lorentz-covariance.

Keywords: nonequilibrium thermodynamics; single-generator bracket; GENERIC; relativistic hydrodynamics; imperfect fluid; thermal conductivity

1 Introduction

The generalized bracket formalism of nonequilibrium thermodynamics introduced by Grmela [1–3] and Beris and Edwards [4, 5] attracted considerable attention in the last decades, since it provides a systematic and solid framework for formulating time-evolution equations for fluids with a complex internal microstructure that are consistent with the fundamental laws of thermodynamics. Coupled, in particular, with a microscopic model describing physics at a lower level, it has been used with remarkable success to address a variety of problems in Soft Matter. We mention, for example, the flow of polymers past a wall [6, 7], the rheology of wormlike micellar solutions [8], the formulation of constitutive equations describing the phase behavior, microstructure, and rheology of unentangled polymer nanocomposite melts [9], the rheology of agglomerating blood [10] and of aggregating particle suspensions [11], and the description of heat and mass transfer in multicomponent systems [12]. The formalism is based on the idea that one can use an appropriate "thermodynamic potential"

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so that the fundamental properties of equilibrium thermodynamics and classical mechanics can be carried over to dynamic, nonequilibrium systems. This fundamental potential is an extended form of the energy of the system, defined as a functional in terms of a small set of independent (hydrodynamic and structural or internal) variables that are assumed to uniquely determine its macroscopic state.

Historically, the development of the Hamiltonian formalism for dissipative systems started with the pioneering works of Kaufman [13], Morrison [14, 15] and Abarbanel et al. [16], and culminated to what is known today as the single-generator bracket formalism of nonequilibrium thermodynamics [17], since the Hamiltonian is the only thermodynamic potential appearing in the master equation of change used to derive the time-evolution equations for the independent variables. In a subsequent study, Grmela and Öttinger proposed using two separate generators of the dynamics [18, 19], the energy and entropy, and this gave rise to the so called GENERIC (= General Equation for the NonEquilibrium Reversible-Irreversible Coupling) or double-generator formalism of nonequilibrium thermodynamics [20].

As already mentioned, the generalized bracket formalism has been widely used to describe transport phenomena in a variety of complex fluids. Naturally, the question arises if it can also be used to describe relativistic fluids which are of paramount importance in astrophysics and cosmology (e.g., in the description of gravitational collapse leading to the formation of neutron stars). For GENERIC, this issue was addressed by Öttinger already since 1998 who showed that the double-generator formalism is fully compatible with the laws of special relativity [21, 22]. Öttinger also discussed how the relativistic equations derived by GENERIC compare with previous theories and reported that the classical theory of Eckart [23], the second-order theory of Israel [24], and the equations of extended irreversible thermodynamics [25] and of kinetic theory [26] do not possess the full nonequilibrium structure characterizing the equations derived from GENERIC, despite that both Israel's theory and extended irreversible thermodynamics are very similar in structure, while kinetic theory can provide the linearized form of the equations. It makes sense therefore to ask the same question for the generalized bracket formalism, and this is exactly the question that we address in this article. Given, in fact, that on the hydrodynamic level, the one- and two-generator frameworks have been shown by Edwards and collaborators to be equivalent [27, 28], the answer to this question is expected to be positive. Our work here will demonstrate that, indeed, the generalized bracket formalism is compatible with special relativity, since with the appropriate choice of the independent variables, of the Hamiltonian, of the Poisson and dissipation brackets, and by appropriately restricting the form of dissipative rates, one can derive a set of manifest Lorentz-covariant equations.

The rest of the paper is structured as follows: In Section 2 we derive first the equations for a nonrelativistic imperfect fluid with heat flow but without viscosity, by defining all building blocks (vector of state variables, Hamiltonian, Poisson bracket, dissipation bracket). Then, in the same Section, we extend the analysis to the case of a relativistic fluid. In Section 3 we examine a fluid with viscosity and dilatational viscosity, and we derive the corresponding nonrelativistic and relativistic equations. In Section 4 we elaborate on the significance of the work and in Section 5 we summarize the most important findings and discuss possible future directions.

2 An inviscid fluid

2.1 Nonrelativistic description

In the context of the generalized bracket formalism of nonequilibrium thermodynamics, dynamic equations are derived by invoking the following general time-evolution equation for an arbitrary functional F [17]:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\} + [F, H] \tag{1}$$

where t denotes the time, H the Hamiltonian of the system expressed in terms of the vector of acceptable state variables, $\{\dots\}$ the Poisson bracket describing conservative (convective) effects to the dynamics, and $[\dots]$ the dissipative bracket accounting for non-conservative phenomena. Coupled time-evolution equations for the relevant dynamical variables are then obtained through a direct comparison of the master equation, Eq. (1), with the expression that results by differentiating $\frac{dF}{dt}$ by parts and using the functional dependence of the Hamiltonian on the vector of state variables.

The two brackets appearing in Eq. (1) must satisfy certain relationships in order for the master equation to be compatible with fundamental physical laws and also have the correct mathematical structure required for an evolution equation. In particular, the Poisson bracket must be bilinear and satisfy the 1st law of thermodynamics for a conservative system; thus, it has to be antisymmetric, $\{F, G\} = -\{G, F\}$, in order to ensure that the total energy of the system is conserved, $\frac{dH}{dt} = \{H, H\} = 0$. It must also satisfy the requirement of zero rate of entropy production for a conservative system, namely $\{S, H\} = 0$, and obey time-structure invariance, i.e., to satisfy the Jacobi identity [29].

The dissipative bracket, on the other hand, must be a linear function of F, must be frame indifferent, and must satisfy the 1st law of thermodynamics, i.e., $\frac{dH}{dt} = 0$, which translates into $[H, H] = 0$. It must also satisfy the 2nd law of thermodynamics, i.e., the requirement of a non-negative rate of entropy production, $\frac{dS}{dt} \geq 0$, which translates into $[S, H] \geq 0$. Moreover, close to equilibrium, the dissipation bracket must have a symmetric structure indicative of the Onsager–Casimir reciprocal relationships [30–32] between the transport coefficients relating fluxes to affinities (the derivatives or the gradient of the Hamiltonian with respect to the dynamical variables). A nice feature of the generalized bracket formalism is that the appropriate expressions for the fluxes arise automatically from the formulation (they come out to depend only on the dissipation bracket) and should not be specified explicitly.

2.1.1 The vector of state variables

The starting point within the bracket formalism is the definition of the set $\mathbf{x}$ of acceptable state variables needed to specify the thermodynamic-hydrodynamic state of the system. This typically contains field densities, including, at a minimum, those needed to specify the system at equilibrium. For example, for a structureless fluid, the set $\mathbf{x}$ contains the mass density of the fluid ρ, the entropy density s, and the momentum density $\mathbf{M}$; thus, we write $\mathbf{x} = \{\rho, \mathbf{M}, s\}$. The next step is to specify the Hamiltonian H, physically identified with an (extended) energy of the system, as a functional (integral over the system volume) of the state variables.

Given the striking similarity of the single-generator formalism with the double-generator formalism, to check the consistency of the generalized bracket formalism with special relativity it makes sense to follow in the present work the path already undertaken by Öttinger in his analysis of the compatibility of the GENERIC formalism with special relativity [21, 22]. Thus, we take the vector $\mathbf{x}$ to be an expanded set of variables consisting of the usual set $\{\rho, \mathbf{M}, s\}$ plus additional vector and/or tensor variables which are covariant and of intensive nature. More specifically, to describe heat flow we introduce a vector $\mathbf{w}$ playing the role of a generalized force whose dynamics leads to a heat flux. We will find that the vector $\mathbf{w}$ is closely related to the space gradient of the temperature of the fluid, and since the derivative with respect to a contravariant vector is of covariant nature, $\mathbf{w}$ will be assumed to be of covariant nature. Similar, to describe viscous effects in Section 3, we will introduce a symmetric, covariant tensor $\mathbf{c}$ playing the role of a generalized mechanical force closely related to the velocity gradient tensor. Overall, for the case of an imperfect fluid without viscosity but with heat flow considered in this Section, the vector of state variables reads $\mathbf{x} = \{\rho(\mathbf{r}, t), \mathbf{M}(\mathbf{r}, t), s(\mathbf{r}, t), \mathbf{w}(\mathbf{r}, t)\}$.

2.1.2 Hamiltonian, Poisson bracket, and the reversible part of transport equations

Given the covariance nature of $\mathbf{w}$, the Poisson bracket for the set $\mathbf{x} = \{\rho, \mathbf{M}, s, \mathbf{w}\}$ has the following form [17, 20]:

$$\begin{aligned}\{F,G\} = &-\int\left[\frac{\delta F}{\delta\rho}\nabla_j\left(\frac{\delta G}{\delta M_j}\rho\right)-\frac{\delta G}{\delta\rho}\nabla_j\left(\frac{\delta F}{\delta M_j}\rho\right)\right]\mathrm{d}V\\ &-\int\left[\frac{\delta F}{\delta M_k}\nabla_j\left(\frac{\delta G}{\delta M_j}M_k\right)-\frac{\delta G}{\delta M_k}\nabla_j\left(\frac{\delta F}{\delta M_j}M_k\right)\right]\mathrm{d}V\\ &-\int\left[\frac{\delta F}{\delta s}\nabla_j\left(\frac{\delta G}{\delta M_j}s\right)-\frac{\delta G}{\delta s}\nabla_j\left(\frac{\delta F}{\delta M_j}s\right)\right]\mathrm{d}V\\ &-\int\left[\frac{\delta F}{\delta w_k}\nabla_k\left(\frac{\delta G}{\delta M_j}\right)w_j-\frac{\delta G}{\delta w_k}\nabla_k\left(\frac{\delta F}{\delta M_j}\right)w_j\right]\mathrm{d}V\\ &-\int\left[\frac{\delta F}{\delta w_k}(\nabla_j w_k)\frac{\delta G}{\delta M_j}-\frac{\delta G}{\delta w_k}(\nabla_j w_k)\frac{\delta F}{\delta M_j}\right]\mathrm{d}V\\ &-\int\left[\frac{\delta F}{\delta s}\nabla_j\left(\frac{\delta G}{\delta w_j}\right)-\frac{\delta G}{\delta s}\nabla_j\left(\frac{\delta F}{\delta w_j}\right)\right]\mathrm{d}V \end{aligned}\tag{2}$$

where the last 3 terms have been chosen so as to give rise to a frame invariant, objective time derivative for the (covariant) vector **w**. Please note that in the above equations and in the following, Einstein's notation implying summation over repeated indices has been tacitly adopted. Also, as it is customary in the field, with Greek indices we will be denoting the four time-space components while with Latin ones we will be denoting only the space components. The above bracket is bilinear, antisymmetric and satisfies the Jacobi identity. In particular, the proof of the Jacobi identity for the terms involving the vector **w** has been worked out by Öttinger [20].

For the Hamiltonian H we assume the following form:

$$\begin{aligned}H(\rho,\mathbf{M},s,\mathbf{w}) &= \int\left(e_{\text{kin}}+u\right)\mathrm{d}V\\ &=\int\left[\frac{|\mathbf{M}(\mathbf{r},t)|^2}{2\rho(\mathbf{r},t)}+u\left(\rho(\mathbf{r},t),s(\mathbf{r},t),\mathbf{w}^2(\mathbf{r},t)\right)\right]\mathrm{d}V,\end{aligned}\tag{3}$$

where e_{kin} denotes the kinetic energy density of the fluid and u the internal energy density. For u we assume the following functional dependence:

$$u = u\left(\rho, s, \mathbf{w}^2\right),\tag{4}$$

which can be regarded as a generalized local thermodynamic relationship. In Eq. (3), we have neglected contributions due to external fields. Then, in the course of the derivation of the equations of motion for the fluid, we need the functional derivatives of the Hamiltonian H with respect to the components of the vector $\mathbf{x}$ of independent variables. Based on Eq. (3), these are:

$$\frac{\delta H}{\delta\mathbf{M}}=\frac{\mathbf{M}}{\rho}=\mathbf{v}\tag{5a}$$

$$\frac{\delta H}{\delta\rho}=-\frac{\mathbf{M}^2}{2\rho^2}+\frac{\partial u}{\partial\rho}\tag{5b}$$

$$\frac{\delta H}{\delta s}=\frac{\partial u}{\partial s}=T\tag{5c}$$

$$\frac{\delta H}{\delta\mathbf{w}}=\mathbf{j}=\sigma\,\mathbf{w}\tag{5d}$$

where we have defined

$$\sigma\equiv 2\frac{\partial u}{\partial\mathbf{w}^2}.\tag{6}$$

Please note that in the above derivatives, ρ, **M** and s are densities of extensive variables but $\mathbf{w}^2$ is by nature an intensive variable.

The dynamic equations of motion are derived by writing the dynamical equation for any functional $F[\rho,\mathbf{M},s,\mathbf{w}]=\int f(\rho,\mathbf{M},s,\mathbf{w})\mathrm{d}V$ in the form $\frac{\mathrm{d}F}{\mathrm{d}t}=\int\left[\frac{\delta F}{\delta\rho}\frac{\partial\rho}{\partial t}+\frac{\delta F}{\delta M_i}\frac{\partial M_i}{\partial t}+\frac{\delta F}{\delta s}\frac{\partial s}{\partial t}+\frac{\delta F}{\delta w_i}\frac{\partial w_i}{\partial t}\right]\mathrm{d}V$ and comparing terms one-by-one with the general evolution equation, Eq. (1), for the Poisson bracket defined above. By doing so, making use of Eqs. (5a)–(5d), defining the thermodynamic pressure p through

$$p = \rho \frac{\partial u}{\partial \rho}\bigg|_{s,\mathbf{w}^2} + s\frac{\partial u}{\partial s}\bigg|_{\rho,\mathbf{w}^2} - u\left(\rho, s, \mathbf{w}^2\right), \tag{7}$$

and combining terms in the momentum equation in an appropriate way, the following set of dynamic equations is derived for such an inviscid fluid:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v}\rho), \tag{8a}$$

$$\frac{\partial \mathbf{M}}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v}\,\mathbf{M}) - \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{P}, \tag{8b}$$

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \left(\mathbf{v}\,s + \mathbf{j}\right), \tag{8c}$$

$$\frac{\partial \mathbf{w}}{\partial t} = -\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}}\mathbf{w} - \boldsymbol{\kappa}^T \cdot \mathbf{w} - \frac{\partial}{\partial \mathbf{r}}T. \tag{8d}$$

In these equations, $\left(\kappa^T\right)_{ij} = \nabla_i v_j$ denotes the *ij*-element of the transpose velocity gradient tensor while $\mathbf{P}$ is the stress tensor defined as

$$\mathbf{P} = p\mathbf{1} + \mathbf{j}\mathbf{w}, \tag{9}$$

with $\mathbf{1}$ denoting the unit tensor of rank 2. Because of Eq. (5d), the tensor $\mathbf{P}$ is symmetric. Equation (8a) is the conservation equation for the mass (the continuity equation), Eq. (8b) is the conservation equation for the momentum (the momentum balance equation), Eq. (8c) is the conservation equation for the entropy, and Eq. (8d) is the time-evolution equation for the thermal vector (the thermal vector equation). Equations (8) are identical to those reported in [21].

2.1.3 Dissipative bracket and the full form of the transport equations

For a system described by the set $\mathbf{x} = \{\rho(\mathbf{r}, t), \mathbf{M}(\mathbf{r}, t), s(\mathbf{r}, t), \mathbf{w}(\mathbf{r}, t)\}$ of hydrodynamic variables, the simplest possible bilinear form for the dissipative bracket is [17]

$$\begin{aligned}[F, G] = &-\int Q_{ijkl}\nabla_i\left(\frac{\delta F}{\delta M_j}\right)\nabla_k\left(\frac{\delta G}{\delta M_l}\right)\mathrm{d}V + \int \frac{1}{T}\frac{\delta F}{\delta s}Q_{ijkl}\nabla_i\left(\frac{\delta G}{\delta M_j}\right)\nabla_k\left(\frac{\delta G}{\delta M_l}\right)\mathrm{d}V \\ &-\int A_{ij}\nabla_i\left(\frac{\delta F}{\delta s}\right)\nabla_j\left(\frac{\delta G}{\delta s}\right)\mathrm{d}V + \int \frac{1}{T}\frac{\delta F}{\delta s}A_{ij}\nabla_i\left(\frac{\delta G}{\delta s}\right)\nabla_j\left(\frac{\delta G}{\delta s}\right)\mathrm{d}V \\ &-\int R_{ij}\frac{\delta F}{\delta w_i}\frac{\delta G}{\delta w_j}\mathrm{d}V + \int \frac{1}{T}\frac{\delta F}{\delta s}R_{ij}\frac{\delta G}{\delta w_i}\frac{\delta G}{\delta w_j}\mathrm{d}V,\end{aligned} \tag{10}$$

where the various phenomenological coefficients should satisfy the following constraints due to the Onsager–Casimir reciprocity principles [17, 30–32]:

$$Q_{ijkl} = Q_{klij}, \quad \forall i, j, k, l \tag{11a}$$

$$A_{ij} = A_{ji}, \quad \forall i, j \tag{11b}$$

$$R_{ij} = R_{ji}, \quad \forall i, j. \tag{11c}$$

Then, the resulting set of transport equations reads

$$\frac{\partial \rho}{\partial t} = -\nabla_j(v_j\rho), \tag{12a}$$

$$\frac{\partial M_i}{\partial t} = -\nabla_j(v_jM_i) - \nabla_jP_{ji} + \nabla_j\left(Q_{jikl}\nabla_k\frac{\delta H}{\delta M_l}\right), \tag{12b}$$

$$\begin{aligned}\frac{\partial s}{\partial t} &= -\nabla_j(v_js + j_j) + \frac{1}{T}Q_{ijkl}\nabla_i\left(\frac{\delta H}{\delta M_j}\right)\nabla_k\left(\frac{\delta H}{\delta M_l}\right) + \frac{1}{T}A_{ij}\nabla_i\left(\frac{\delta H}{\delta s}\right)\nabla_j\left(\frac{\delta H}{\delta s}\right)\\ &\quad + \nabla_i\left(A_{ij}\nabla_j\left(\frac{\delta H}{\delta s}\right)\right) + \frac{1}{T}R_{ij}\frac{\delta H}{\delta w_i}\frac{\delta H}{\delta w_j},\end{aligned} \tag{12c}$$

$$\frac{\partial w_i}{\partial t} = -v_j(\nabla_jw_i) - (\nabla_iv_j)w_j - \nabla_iT - R_{ij}\frac{\delta H}{\delta w_j}. \tag{12d}$$

For an isotropic fluid, the three phenomenological coefficients should have the following form [17]:

$$\begin{aligned}A_{ij} &= A_{00}\delta_{ij}, \quad \forall i,j\\ Q_{ijkl} &= \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \kappa'\delta_{ij}\delta_{kl}, \quad \forall i,j,k,l\\ R_{ij} &= R_{00}\delta_{ij}, \quad \forall i,j,\end{aligned} \tag{13}$$

with μ and κ' having units of viscosity. Related to them is the bulk viscosity defined as $\kappa = \kappa' + \frac{2}{3}\mu$. The above expressions result in the following set of time-evolution equations:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}\rho), \tag{14a}$$

$$\frac{\partial \mathbf{M}}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}\,\mathbf{M}) - \frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{P} + \frac{\partial}{\partial \mathbf{r}}\cdot\boldsymbol{\sigma}, \tag{14b}$$

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}\,s + \mathbf{j}) + \frac{1}{T}\boldsymbol{\sigma}:\boldsymbol{\kappa}^T + \frac{A_{00}}{T}\frac{\partial}{\partial \mathbf{r}}T\cdot\frac{\partial}{\partial \mathbf{r}}T + \frac{\partial}{\partial \mathbf{r}}\cdot\left(A_{00}\frac{\partial}{\partial \mathbf{r}}T\right) + \frac{1}{T}R_{00}\,\mathbf{j}^2, \tag{14c}$$

$$\frac{\partial \mathbf{w}}{\partial t} = -\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}\mathbf{w} - \boldsymbol{\kappa}^T\cdot\mathbf{w} - \frac{\partial}{\partial \mathbf{r}}T - R_{00}\,\mathbf{j}, \tag{14d}$$

where $\boldsymbol{\sigma}$ is the extra stress tensor given by

$$\boldsymbol{\sigma} = \mu(\boldsymbol{\kappa}^T + \boldsymbol{\kappa}) + \kappa'\left(\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{v}\right)\mathbf{1}. \tag{15}$$

In the next Section, we will focus on the relativistic case where no terms proportional to **Q** and **A** are allowed [21]. In this case, the transport equations reduce to

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}\rho), \tag{16a}$$

$$\frac{\partial \mathbf{M}}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}\,\mathbf{M}) - \frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{P}, \tag{16b}$$

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}\,s + \mathbf{j}) + \frac{1}{T}R_{00}\,\mathbf{j}^2, \tag{16c}$$

$$\frac{\partial \mathbf{w}}{\partial t} = -\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}\mathbf{w} - \boldsymbol{\kappa}^T\cdot\mathbf{w} - \frac{\partial}{\partial \mathbf{r}}T - R_{00}\,\mathbf{j}. \tag{16d}$$

From Eq. (16c), in particular, we can identify the thermal conductivity as $\lambda = \frac{1}{T}R_{00}$.

2.2 Relativistic formulation

2.2.1 Definitions of fluid properties and some important thermodynamic considerations

Due to the increased density of the fluid from the contraction of length in the direction of flow, for the rest mass density we take

$$\rho = \gamma\, \rho_f, \tag{17}$$

where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{18}$$

is the Lorentz factor, with c being the speed of light. In Eq. (17) but also everywhere in the following analysis, the subscript "f" will be used to denote properties of the fluid evaluated in its local (comoving) frame of reference. The next step is to introduce the Minkowski tensor for raising and lowering indices:

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}$$

and define the fluid velocity. Despite that there are various possibilities of defining the local fluid velocity, following Öttinger [21, 22] we will prefer Eckart's definition [23] according to which, in the corresponding reference frame of the fluid, the time-like Lorentz vector u^μ of the fluid velocity becomes $u_f^0 = 1$ and $u_f^i = 0$ $(i = 1, 2, 3)$. In the fixed Cartesian frame, we work with the dimensionless velocity four-vector u^μ defined in terms of the components v_i of the velocity vector $\mathbf{v}$ through

$$u^0 = \gamma, \quad u^i = \gamma \frac{v_i}{c} \tag{20}$$

so that $u_\mu u^\mu = -1$. We also introduce the four-vector of partial derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$ $(\mu = 0, 1, 2, 3)$ where $x^0 = -x_0 = ct$ is the time coordinate and $x^i = x_i$ $(i = 1, 2, 3)$ are the three space coordinates (the Cartesian components of the position vector $\mathbf{r}$).

Of key importance for the rest of the analysis is the definition of the energy-momentum tensor. To this, and following [21], we assume that this tensor has the following general form:

$$T^{\mu\nu} = \left(\rho_f c^2 + u_f - u_\alpha P^{\alpha\beta} u_\beta\right) u^\mu u^\nu + P^{\mu\nu}, \tag{21}$$

where, inspired from Eq. (9) of the nonrelativistic case, the stress tensor is taken to be:

$$P^{\mu\nu} = p_f \eta^{\mu\nu} + \sigma_f w^\mu w^\nu. \tag{22}$$

This implies that we have elevated the 3-d vector $\mathbf{w}$ to a four-vector w^μ parameterized by its time component w^0 and its three Cartesian components w_i. The time component w^0 of the vector $\mathbf{w}$ will be defined a little later by postulating a generalized (thermodynamic-hydrodynamic) Euler equation for the relativistic fluid. In Eq. (21), u_f is the internal energy of the fluid in the comoving frame of reference, for which we assume the following functional dependence:

$$u_f = u_f\left(\rho_f, s_f, \mathbf{w}_f^{\,2}\right), \tag{23}$$

with s_f denoting the entropy density of the fluid (in the comoving frame of reference). Similar, p_f is the thermodynamic pressure of the fluid in the comoving frame of reference. How $\mathbf{w}_f$ is related to $\mathbf{w}$ is discussed by Öttinger

[21]. In analogy with the nonrelativistic case, we also introduce the following (relativistic) fluid properties in the comoving frame:

$$\mu_f = \frac{\partial u_f}{\partial \rho_f}, \tag{24a}$$

$$T_f = \frac{\partial u_f}{\partial s_f}, \tag{24b}$$

$$\sigma_f = 2\frac{\partial u_f}{\partial {\mathbf{w}_f}^2}, \tag{24c}$$

and we further assume the following thermodynamic Euler equation:

$$u_f = T_f s_f - p_f + \mu_f \rho_f. \tag{25}$$

From Eqs. (21) and (22), the expression for the energy-momentum tensor becomes

$$T^{\mu\nu} = \left(\rho_f c^2 + u_f + p_f + \sigma_f \left(u_\alpha w^\alpha\right)^2\right) u^\mu u^\nu + P^{\mu\nu}. \tag{26}$$

From Eq. (21) for $T^{\mu\nu}$, we also identify the momentum vector components ($M_i \equiv \frac{1}{c} T^{0i}$) as

$$M_i = \left(T^{00} - P^{00}\right)\frac{v_i}{c^2} + \frac{1}{c}P^{0i}, \tag{27}$$

or, equivalently,

$$\mathbf{M} = \left(\rho_f c^2 + u_f - u_\alpha P^{\alpha\beta} u_\beta\right)\gamma^2 \frac{\mathbf{v}}{c^2} + \sigma_f w^0 \frac{\mathbf{w}}{c}. \tag{28}$$

By noticing that

$$T^{00} = \left(\rho_f c^2 + u_f - u_\alpha P^{\alpha\beta} u_\beta\right)\gamma^2 - p_f + \sigma_f w^0 w^0, \tag{29}$$

the momentum vector can also be written as

$$\mathbf{M} = \left(T^{00} + p_f - \sigma_f w_0^2\right)\frac{\mathbf{v}}{c^2} - \sigma_f w_0 \frac{\mathbf{w}}{c}. \tag{30}$$

Since $\rho_f c^2$ is the contribution of the mass to the energy, the rest of the terms in Eq. (29) should represent the *total internal energy* density of the fluid (the sum of internal and kinetic energy densities).

To develop suitable transport equations for the relativistic fluid, we take the vector of state variables to be again $\mathbf{x} = \{\rho, \mathbf{M}, s, \mathbf{w}\}$ as for the nonrelativistic one. Then, from the energy-momentum tensor, the Hamiltonian (total energy) of the relativistic fluid is

$$H(\mathbf{x}) = \int T^{00} \mathrm{d}^3 r; \tag{31}$$

thus, the functional derivatives $\frac{\delta H}{\delta \mathbf{x}}$ that are needed in the construction of the Poisson and dissipative brackets can be calculated through

$$\frac{\delta H}{\delta \mathbf{x}} = \frac{\partial T^{00}}{\partial \mathbf{x}}. \tag{32}$$

For the vector $\mathbf{x} = \{\rho, \mathbf{M}, s, \mathbf{w}\}$ defined above, these derivatives have been obtained by Öttinger in [21] and read

$$\frac{\delta H}{\delta \rho} = \frac{c^2 + \mu_f}{\gamma}, \tag{33a}$$

$$\frac{\delta H}{\delta \mathbf{M}} = \mathbf{v}, \tag{33b}$$

$$\frac{\delta H}{\delta s} = \frac{T_f}{\gamma}, \tag{33c}$$

$$\frac{\delta H}{\delta \mathbf{w}} = \mathbf{j}, \tag{33d}$$

where now the heat flux vector is given not by Eq. (5d) but by

$$\mathbf{j} = \sigma_f \mathbf{w} + \sigma_f w_0 \frac{\mathbf{v}}{c}. \tag{34}$$

With the help of Eq. (33), we can also fix now the time-like component of the four-vector w^μ as well as the fluid entropy in the comoving frame. To this, we demand the following generalized (relativistic hydrodynamic-thermodynamic) Euler equation at the level of the total energy of the relativistic fluid:

$$\rho \frac{\delta H}{\delta \rho} + \mathbf{M} \cdot \frac{\delta H}{\delta \mathbf{M}} + s \frac{\delta H}{\delta s} = p_f + T^{00}. \tag{35}$$

In Appendix A we examine under what conditions such an equation applies and we find that Eq. (35) is identically satisfied if: (a) we chose w^0 to be such that

$$w_\mu u^\mu = -\frac{T_f}{c} \tag{36}$$

and (b) take the entropy density to be

$$s = \gamma s_f + \frac{\sigma_f}{c}\left(w^0 - \frac{T_f}{c}\gamma\right). \tag{37}$$

These were the missing building blocks before applying the generalized bracket formalism to derive the relativistic hydrodynamic equations for a relativistic fluid described by the set $\mathbf{x} = \{\rho, \mathbf{M}, s, \mathbf{w}\}$. In fact, Eq. (37) suggests the following expression for the four-vector entropy of the relativistic fluid:

$$S^\mu = s_f u^\mu + \frac{J^\mu}{c}, \tag{38}$$

with

$$J^\mu = \sigma_f\left(w^\mu - \frac{T_f}{c}u^\mu\right). \tag{39}$$

This is a beautiful result, since it also indicates how the flux $\mathbf{j}$ is elevated from a 3-d vector to a four-vector. Indeed, from Eq. (34) we find that

$$\mathbf{j} = \sigma_f\left(\mathbf{1} - \frac{\mathbf{v}\,\mathbf{v}}{c^2}\right) \cdot \left(\mathbf{w} - \gamma T_f \frac{\mathbf{v}}{c^2}\right), \tag{40a}$$

implying

$$\mathbf{j} = \mathbf{J} - \frac{J^0}{c}\mathbf{v}, \tag{40b}$$

which is fully compatible with Eq. (39).

Öttinger [21] has indicated several useful equations satisfied by the four-vectors w^μ, u^μ, J^μ and S^μ. In particular, for the following analysis, we note that $u_\mu J^\mu = 0$. With the help of Eq. (36), we also note that

$$T^{00} = \left(\rho_f c^2 + u_f + p_f + \sigma_f \frac{T_f^2}{c^2}\right)\gamma^2 - p_f + \sigma_f w_0^2. \tag{41}$$

Moreover, from Eq. (33c) we understand that $T = \frac{T_f}{\gamma}$, implying that $T_f > T$.

2.2.2 Poisson bracket and the corresponding reversible part of the transport equations

We are now in a position to write down the dynamic equations in the comoving frame. To this, we assume exactly the same form of the Poisson bracket as before, Eq. (2), and we make again use of the following generalized thermodynamic identity (a direct result of the generalized Euler equation expressed by Eq. (35) and the fact that the total energy density is T^{00}):

$$\rho \nabla_i \frac{\delta H}{\delta \rho} + M_j \nabla_i \frac{\delta H}{\delta M_j} + s \nabla_i \frac{\delta H}{\delta s} = \nabla_i p_f + j_j \nabla_i w_j. \tag{42}$$

Then, the reversible part of the dynamic equations comes out to be

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v}\rho), \tag{43a}$$

$$\frac{\partial \mathbf{M}}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v}\,\mathbf{M}) - \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{P}, \tag{43b}$$

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \left(\mathbf{v}\, s + \mathbf{j}\right), \tag{43c}$$

$$\frac{\partial \mathbf{w}}{\partial t} = -\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} \mathbf{w} - \boldsymbol{\kappa}^T \cdot \mathbf{w} - \frac{\partial}{\partial \mathbf{r}} \left(\frac{T_f}{\gamma} \right), \tag{43d}$$

where $\mathbf{P}$ is the stress tensor defined as

$$\mathbf{P} = p_f \mathbf{1} + \mathbf{j}\,\mathbf{w}. \tag{44}$$

Interestingly enough, and despite the totally different expressions of $\mathbf{j}$ in the relativistic and nonrelativistic cases, the corresponding equation between $\mathbf{P}$ and $\mathbf{j}$ remains the same.

The explicit Lorentz-covariant form of the above transport equations is

$$\partial_\mu \left(\rho_f u^\mu\right) = 0, \tag{45a}$$

$$\partial_\mu \left[\left(\rho_f c^2 + u_f - u_\alpha P^{\alpha\beta} u_\beta\right) u^\mu u^\nu + P^{\mu\nu}\right] = 0, \tag{45b}$$

$$\partial_\mu \left(s_f u^\mu + \frac{J^\mu}{c} \right) = 0, \tag{45c}$$

$$u^\mu \left(\partial_\nu w_\mu - \partial_\mu w_\nu\right) = 0. \tag{45d}$$

2.2.3 Dissipative bracket and full form of transport equations

To complete the above set of equations, we need to add the dissipative contributions. Given that we make use of the same set of structural variables as in the nonrelativistic case, we start by using the same general form of the dissipative bracket as in the nonrelativistic case, Eq. (13), without the terms proportional to tensors $\mathbf{Q}$ and $\mathbf{A}$, and with the understanding that, due to the Onsager–Casimir reciprocity principles, the tensor $\mathbf{R}$ should be again symmetric, see Eq. (11c):

$$\begin{aligned}[F, G] = &-\int A_1 w_i \frac{\delta F}{\delta w_i} \frac{\delta G}{\delta s} \mathrm{d}V + \int \frac{1}{T} \frac{\delta F}{\delta s} A_1 w_i \frac{\delta G}{\delta w_i} \frac{\delta G}{\delta s} \mathrm{d}V \\ &- \int R_{ij} \frac{\delta F}{\delta w_i} \frac{\delta G}{\delta w_j} \mathrm{d}V + \int \frac{1}{T} \frac{\delta F}{\delta s} R_{ij} \frac{\delta G}{\delta w_i} \frac{\delta G}{\delta w_j} \mathrm{d}V.\end{aligned} \tag{46}$$

However, in the relativistic case, the tensor $\mathbf{R}$ should not have to be isotropic. In fact, $\mathbf{R}$ has to be defined such that the resulting equations are manifest Lorentz-covariant, which places considerable constraints on its admissible forms. With Eq. (46) for the dissipation bracket, the resulting set of transport equations for the entropy and thermal vector (the balance equations for the mass and momentum densities remain the same) read:

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \left(\mathbf{v}\, s + \mathbf{j}\right) + A_1 \mathbf{w} \cdot \mathbf{j} + \frac{1}{T} \mathbf{R} \cdot \mathbf{j} \cdot \mathbf{j}, \tag{47a}$$

$$\frac{\partial \mathbf{w}}{\partial t} = -\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} \mathbf{w} - \boldsymbol{\kappa}^T \cdot \mathbf{w} - \frac{\partial}{\partial \mathbf{r}} \left(\frac{T_f}{\gamma} \right) - A_1 T \mathbf{w} - \mathbf{R} \cdot \mathbf{j}. \tag{47b}$$

The final step is to determine a particular form of the matrix $\mathbf{R}$ for which a covariant set of transport equations arises. One way to achieve this is to note that from the second equality in Eq. (40), the vector $\mathbf{j}$ can be expressed as

$$\mathbf{j} = \mathbf{F} \cdot \mathbf{J}, \tag{48}$$

where

$$\mathbf{F} = \mathbf{1} - \frac{\mathbf{v}\,\mathbf{v}}{c^2}, \tag{49}$$

and $\mathbf{J}$ is given from Eq. (39). Then, we can require that the matrix $\mathbf{R}$ satisfies: $R_{\alpha\beta}F_{\alpha\gamma} \equiv A_2 \cdot \delta_{\beta\gamma}$, where A_2 a rate constant. That is, we assume that the matrix $\mathbf{R}$ is proportional to the inverse of the matrix $\mathbf{F}$:

$$\mathbf{R} \equiv A_2 \cdot \mathbf{F}^{-1}. \tag{50}$$

Then, one can show that

$$\mathbf{R} = A_2\left(\mathbf{1} + \gamma^2 \frac{\mathbf{v}\,\mathbf{v}}{c^2}\right), \tag{51}$$

which is exactly the corresponding entry in the friction matrix proposed by Öttinger in Ref. [21]. According to Eq. (51), the friction matrix $\mathbf{R}$ depends on the velocity field, thus it is inherently anisotropic which significantly differentiates it from the nonrelativistic case. Then, the governing equations for the entropy and thermal vector become

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \left(\mathbf{v}\,s + \mathbf{j}\right) + A_1 \mathbf{w} \cdot \mathbf{j} + \frac{A_2}{T}\mathbf{J} \cdot \mathbf{j}, \tag{52a}$$

$$\frac{\partial \mathbf{w}}{\partial t} = -\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}}\mathbf{w} - \boldsymbol{\kappa}^T \cdot \mathbf{w} - \frac{\partial}{\partial \mathbf{r}}\left(\frac{T_f}{\gamma}\right) - A_1 T\mathbf{w} - A_2 \mathbf{J}. \tag{52b}$$

The next step is to check if these can be cast in a Lorentz-covariant form. For the 2nd entropy production term on the right-hand side of Eq. (52a), we notice that

$$\mathbf{J} \cdot \mathbf{j} = J_\mu J^\mu, \tag{53}$$

which is manifest Lorentz-covariant. For the 1st entropy production term, on the other hand, we find

$$w_i j_i = w_\mu J^\mu - \frac{1}{c}\frac{T_f}{\gamma} J^0, \tag{54}$$

which is clearly noncovariant. Thus, we must take $A_1 = 0$. Consequently, the final covariant form of the four dynamic equations [Eq. (43a) for the density, Eq. (43b) for the energy-momentum tensor, Eq. (52a) for the entropy density, and Eq. (52b) for the thermal vector] reads:

$$\partial_\mu\left(\rho_f u^\mu\right) = 0, \tag{55a}$$

$$\partial_\mu\left[\left(\rho_f c^2 + u_f - u_\alpha P^{\alpha\beta} u_\beta\right) u^\mu u^\nu + P^{\mu\nu}\right] = 0, \tag{55b}$$

$$\partial_\mu\left(s_f u^\mu + \frac{J^\mu}{c}\right) = \frac{A_2}{T} J_\mu J^\mu, \tag{55c}$$

$$u^\mu\left(\partial_\nu w_\mu - \partial_\nu w_\mu\right) = -A_2\left(w_\nu - \frac{T_f}{c} u_\nu\right)\left[= A_2\left(\eta_{\nu\lambda} + u_\nu u_\lambda\right) w^\lambda\right]. \tag{55d}$$

The set of relativistic equations, Eq. (55), is identical to that derived by Öttinger from GENERIC [21]; however, we have to keep in mind that they have been obtained by choosing the phenomenological friction matrix $\mathbf{R}$ to satisfy Eq. (50). Thus, in principle, one could think of other choices of $\mathbf{R}$ that could render the time-evolution equations for the entropy and the thermal vector covariant, and this is something worth pursuing further in the future. For the rest of the analysis, we also note the following equations (see also [21]):

$$w_\mu J^\mu = \sigma_f \mathbf{w}_f^2, \tag{56}$$

$$J_\mu J^\mu = \sigma_f w_\mu J^\mu = \sigma_f^2 \mathbf{w}_f^2, \tag{57}$$

$$\mathbf{w}_f = \left(\mathbf{1} - \frac{\gamma}{\gamma + 1}\frac{\mathbf{v}\,\mathbf{v}}{c^2}\right) \cdot \left(\mathbf{w} - \gamma T_f \frac{\mathbf{v}}{c^2}\right). \tag{58}$$

3 A viscous fluid

Having succeeded in deriving a set of relativistic equations for an imperfect fluid with heat flow that possess the full structure of the single-generator bracket formalism, the next step is to consider the case of an imperfect fluid with viscosity and dilatational viscosity. Again, we start by introducing the proper set of variables. Following Öttinger [22], we now add in the vector of state variables a tensor variable **c** which, as will see in the course of the analysis, is intimately connected with the velocity gradient tensor and thus with the momentum flux. As in Section 2, we will consider first the nonrelativistic case and then we will proceed to the relativistic description. Considering the nonrelativistic case first will be helpful in several respects: (a) it will lead to equations that can reproduce hydrodynamics with the correct dissipation terms, (b) it will allow us to guess the proper form of the stress tensor to include in the relativistic description, and (c) it will provide information concerning the most essential couplings between momentum flow and heat flow. Moreover, the analysis will help us define fluid properties which will be identified next with the fundamental transport properties of the fluid (thermal conductivity, viscosity, and dilatational viscosity), which of course need to be non-negative.

3.1 Nonrelativistic formulation

3.1.1 The vector of state variables

We take the vector of state variables to be $\mathbf{x} = \{\rho, \mathbf{M}, s, \mathbf{w}, \mathbf{c}\}$ where **c** is a symmetric covariant tensor. However, and as will become more apparent in the following sections, when we will elevate this tensor to a covariant tensor $c_{\mu\nu}$, eventually we will have to work not with the tensor **c** but with the tensor $\mathbf{c} - \mathbf{1}$. Thus, in the following, the new structural variable will be (not the covariant tensor **c** but) the covariant tensor $\mathbf{c} - \mathbf{1}$.

3.1.2 Hamiltonian, Poisson bracket, and the reversible part of transport equations

Given the covariant nature of $\mathbf{c} - \mathbf{1}$, the Poisson bracket for the set **x** just defined in Section 3.1.1 has the following form:

$$\begin{aligned}\{F,G\} = \{F,G\}_{\text{inv}} &- \int \left[\frac{\delta F}{\delta c_{ij}} \nabla_k(c_{ij}) \frac{\delta G}{\delta M_k} - \frac{\delta G}{\delta c_{ij}} \nabla_k(c_{ij}) \frac{\delta F}{\delta M_k}\right] \mathrm{d}V \\ &+ \int (c_{ik} - \delta_{ik}) \left[\frac{\delta G}{\delta c_{ij}} \nabla_j\left(\frac{\delta F}{\delta M_k}\right) - \frac{\delta F}{\delta c_{ij}} \nabla_j\left(\frac{\delta G}{\delta M_k}\right)\right] \mathrm{d}V \\ &+ \int (c_{kj} - \delta_{kj}) \left[\frac{\delta G}{\delta c_{ij}} \nabla_i\left(\frac{\delta F}{\delta M_k}\right) - \frac{\delta F}{\delta c_{ij}} \nabla_i\left(\frac{\delta G}{\delta M_k}\right)\right] \mathrm{d}V,\end{aligned} \tag{59}$$

where $\{F,G\}_{\text{inv}}$ corresponds to the inviscid case and is given by Eq. (2). The above bracket is bilinear, antisymmetric and satisfies the Jacobi identity. The corresponding expression for the Hamiltonian H reads

$$\begin{aligned}H(\rho, \mathbf{M}, s, \mathbf{w}, \mathbf{c}) &= \int (e_{\text{kin}} + u) \mathrm{d}V \\ &= \int \left[\frac{|\mathbf{M}(\mathbf{r},t)|^2}{2\rho(\mathbf{r},t)} + u(\rho(\mathbf{r},t), s(\mathbf{r},t), \mathbf{w}(\mathbf{r},t), \mathbf{c}(\mathbf{r},t))\right] \mathrm{d}V,\end{aligned} \tag{60}$$

with the dependence of the internal energy u on the vector **w** and tensor **c** coming through the following invariant (scalar) quantities:

$$I_1 = \mathrm{tr}(\mathbf{c}), \quad I_2 = \mathrm{tr}(\mathbf{c}^2), \quad I_3 = \mathrm{tr}(\mathbf{c}^3), \quad I_4 = \mathbf{w}^2, \quad I_5 = \mathbf{w} \cdot \mathbf{c} \cdot \mathbf{w}, \quad I_6 = \mathbf{w} \cdot \mathbf{c}^2 \cdot \mathbf{w}. \tag{61}$$

Thus, for the internal energy u we assume a general thermodynamic relationship of the form:

$$u = u(\rho, s, I_1, I_2, I_3, I_4, I_5, I_6), \tag{62}$$

from which the following fluid properties are defined:

$$\mu = \frac{\partial u}{\partial \rho}, \tag{63a}$$

$$T = \frac{\partial u}{\partial s}, \tag{63b}$$

$$\sigma_1 = 2\frac{\partial u}{\partial I_1}, \tag{63c}$$

$$\sigma_2 = 4\frac{\partial u}{\partial I_2}, \tag{63d}$$

$$\sigma_3 = 6\frac{\partial u}{\partial I_3}, \tag{63e}$$

$$\sigma_4 = 4\frac{\partial u}{\partial I_4}, \tag{63f}$$

$$\sigma_5 = 2\frac{\partial u}{\partial I_5}, \tag{63g}$$

$$\sigma_6 = 2\frac{\partial u}{\partial I_6}. \tag{63h}$$

Based on these, the functional derivatives of the Hamiltonian with respect to the independent variables $\mathbf{x} = \{\rho, \mathbf{M}, s, \mathbf{w}, \mathbf{c}\}$ that are needed in the course of the derivation of the equations of motion turn out to be

$$\frac{\delta H}{\delta \mathbf{M}} = \frac{\mathbf{M}}{\rho} = \mathbf{v}, \tag{64a}$$

$$\frac{\delta H}{\delta \rho} = -\frac{\mathbf{M}^2}{2\rho^2} + \frac{\partial u}{\partial \rho}, \tag{64b}$$

$$\frac{\delta H}{\delta s} = \frac{\partial u}{\partial s} = T, \tag{64c}$$

$$\frac{\delta H}{\delta \mathbf{w}} = \frac{\partial u}{\partial \mathbf{w}} \equiv \mathbf{j} = \left(\sigma_4 \mathbf{1} + \sigma_5\,\mathbf{c} + \sigma_6 \mathbf{c}^2\right) \cdot \mathbf{w}, \tag{64d}$$

$$\frac{\delta H}{\delta \mathbf{c}} = \frac{\partial u}{\partial \mathbf{c}} \equiv \boldsymbol{\tau} = \frac{1}{2}\sigma_1 \mathbf{1} + \frac{1}{2}\sigma_2 \mathbf{c} + \frac{1}{2}\sigma_3 \mathbf{c}\cdot\mathbf{c} + \frac{1}{2}\sigma_5 \mathbf{ww} + \frac{1}{2}\sigma_6 (\mathbf{ww}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{ww}). \tag{64e}$$

In writing down, in particular, the last term in Eq. (64e), we have used that the tensor $\mathbf{c}$ is symmetric. We also note that in the above derivatives, ρ, $\mathbf{M}$ and s are densities of extensive variables but $\mathbf{w}$ and $\mathbf{c}$ are by nature intensive variables.

By writing the dynamical equations for any functional $F[\rho, \mathbf{M}, s, \mathbf{w}, \mathbf{c}] = \int f(\rho, \mathbf{M}, s, \mathbf{w}, \mathbf{c})\mathrm{d}V$ in the form $\frac{\mathrm{d}F}{\mathrm{d}t} = \int \left[\frac{\delta F}{\delta \rho}\frac{\partial \rho}{\partial t} + \frac{\delta F}{\delta M_i}\frac{\partial M_i}{\partial t} + \frac{\delta F}{\delta s}\frac{\partial s}{\partial t} + \frac{\delta F}{\delta w_i}\frac{\partial w_i}{\partial t} + \frac{\delta F}{\delta c_{ij}}\frac{\partial c_{ij}}{\partial t}\right]\mathrm{d}V$, comparing terms one-by-one with the general evolution equation, Eq. (1), for the Poisson bracket defined by Eq. (59), introducing the thermodynamic pressure through

$$p = \rho \left.\frac{\partial u}{\partial \rho}\right|_{s,\mathbf{w},\mathbf{c}} + s\left.\frac{\partial u}{\partial s}\right|_{\rho,\mathbf{w},\mathbf{c}} - u \tag{65}$$

which helps simplify the momentum equation, and defining the stress tensor $\mathbf{P}$ through

$$\mathbf{P} = p\mathbf{1} + \mathbf{jw} + 2\boldsymbol{\tau}\cdot(\mathbf{c} - \mathbf{1}), \tag{66a}$$

(where we have used again that $\mathbf{c}$ is symmetric) with $\mathbf{1}$ being the unit tensor of rank 2, the following equations of motion are obtained:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}\rho), \tag{67a}$$

$$\frac{\partial \mathbf{M}}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}\,\mathbf{M}) - \frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{P}, \tag{67b}$$

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \left(\mathbf{v}\, s + \mathbf{j}\right), \tag{67c}$$

$$\frac{\partial \mathbf{w}}{\partial t} = -\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}}\mathbf{w} - \boldsymbol{\kappa}^T \cdot \mathbf{w} - \frac{\partial}{\partial \mathbf{r}}T, \tag{67d}$$

$$\frac{\partial \mathbf{c}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{c} + \mathbf{c} \cdot \boldsymbol{\kappa} + \boldsymbol{\kappa}^T \cdot \mathbf{c} = \boldsymbol{\kappa} + \boldsymbol{\kappa}^T. \tag{67e}$$

By further substituting Eqs. (64d) and (64e) for $\mathbf{j}$ and $\boldsymbol{\tau}$, respectively, into Eq. (66), the following final expression for the stress tensor is obtained:

$$\begin{aligned} \mathbf{P} = p\mathbf{1} + \left(\sigma_1 \mathbf{1} + \sigma_2 \mathbf{c} + \sigma_3 \mathbf{c}^2\right) \cdot (\mathbf{c} - \mathbf{1}) + \left(\sigma_4 - \sigma_5\right)\mathbf{ww} + \left(\sigma_5 - \sigma_6\right)(\mathbf{ww} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{ww}) \\ + \sigma_6\left(\mathbf{ww} \cdot \mathbf{c}^2 + \mathbf{c} \cdot \mathbf{ww} \cdot \mathbf{c} + \mathbf{c}^2 \cdot \mathbf{ww}\right). \end{aligned} \tag{68}$$

3.1.3 Dissipative bracket and the full form of the transport equations

For the dissipative bracket, we can use the following rather general form:

$$\begin{aligned} [F, G] = &-\int A_1 w_i \frac{\delta F}{\delta w_i}\frac{\delta G}{\delta s}\mathrm{d}V + \int \frac{1}{T}\frac{\delta F}{\delta s} A_1 w_i \frac{\delta G}{\delta w_i}\frac{\delta G}{\delta s}\mathrm{d}V \\ &- \int R_{ij}\frac{\delta F}{\delta w_i}\frac{\delta G}{\delta w_j}\mathrm{d}V + \int \frac{1}{T}\frac{\delta F}{\delta s} R_{ij}\frac{\delta G}{\delta w_i}\frac{\delta G}{\delta w_j}\mathrm{d}V \\ &- \int A_2\left(c_{ij} - \delta_{ij}\right)\frac{\delta F}{\delta c_{ij}}\frac{\delta G}{\delta s}\mathrm{d}V + \int \frac{1}{T}\frac{\delta F}{\delta s}A_2\left(c_{ij} - \delta_{ij}\right)\frac{\delta G}{\delta c_{ij}}\frac{\delta G}{\delta s}\mathrm{d}V \\ &- \int Q_{ijkl}\frac{\delta F}{\delta c_{ij}}\frac{\delta G}{\delta c_{kl}}\mathrm{d}V + \int \frac{1}{T}\frac{\delta F}{\delta s}Q_{ijkl}\frac{\delta G}{\delta c_{ij}}\frac{\delta G}{\delta c_{kl}}\mathrm{d}V, \end{aligned} \tag{69}$$

but to keep the equations relatively simple, we will choose $\mathbf{Q} = \mathbf{0}$ in the following. Moreover, we can break the tensor $\mathbf{c}$ into its traceless $\hat{\mathbf{c}}$ and isotropic $\bar{\mathbf{c}}$ parts, namely, $\mathbf{c} = \bar{\mathbf{c}} + \hat{\mathbf{c}}$, with $\bar{\mathbf{c}} = \frac{1}{3}\mathrm{tr}(\mathbf{c})\mathbf{1}$ and $\hat{\mathbf{c}} = \mathbf{c} - \frac{1}{3}\mathrm{tr}(\mathbf{c})\mathbf{1}$, and assume different relaxation rates for the two components in order to distinguish between shear and dilatational effects. Thus, we assume that

$$\begin{aligned} [F, G] = &-\int A_1 w_i \frac{\delta F}{\delta w_i}\frac{\delta G}{\delta s}\mathrm{d}V + \int \frac{1}{T}\frac{\delta F}{\delta s} A_1 w_i \frac{\delta G}{\delta w_i}\frac{\delta G}{\delta s}\mathrm{d}V \\ &- \int R_{ij}\frac{\delta F}{\delta w_i}\frac{\delta G}{\delta w_j}\mathrm{d}V + \int \frac{1}{T}\frac{\delta F}{\delta s} R_{ij}\frac{\delta G}{\delta w_i}\frac{\delta G}{\delta w_j}\mathrm{d}V \\ &- \int \left(A_{21}\left(\bar{c}_{ij} - \delta_{ij}\right) + A_{22}\left(\hat{c}_{ij} - \delta_{ij}\right)\right)\frac{\delta F}{\delta c_{ij}}\frac{\delta G}{\delta s}\mathrm{d}V \\ &+ \int \frac{1}{T}\frac{\delta F}{\delta s}\left(A_{21}\left(\bar{c}_{ij} - \delta_{ij}\right) + A_{22}\left(\hat{c}_{ij} - \delta_{ij}\right)\right)\frac{\delta G}{\delta c_{ij}}\frac{\delta G}{\delta s}\mathrm{d}V. \end{aligned} \tag{70}$$

By computing the extra contributions and adding them to Eq. (67), the following full set of dynamic equations is derived:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v}\rho), \tag{71a}$$

$$\frac{\partial \mathbf{M}}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v}\,\mathbf{M}) - \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{P}, \tag{71b}$$

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \left(\mathbf{v}\, s + \mathbf{j}\right) + A_1 \mathbf{w} \cdot \mathbf{j} + \frac{1}{T}\mathbf{R}\colon \mathbf{jj} + \left(A_{21}\bar{\mathbf{c}} + A_{22}\hat{\mathbf{c}}\right)\colon \boldsymbol{\tau}, \tag{71c}$$

$$\frac{\partial \mathbf{w}}{\partial t} = -\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}}\mathbf{w} - \boldsymbol{\kappa}^T \cdot \mathbf{w} - \frac{\partial}{\partial \mathbf{r}}T - A_1 \cdot T \cdot \mathbf{w} - \mathbf{R} \cdot \mathbf{j}, \tag{71d}$$

$$\frac{\partial \mathbf{c}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{c} + \mathbf{c} \cdot \boldsymbol{\kappa} + \boldsymbol{\kappa}^T \cdot \mathbf{c} = \boldsymbol{\kappa} + \boldsymbol{\kappa}^T - T \cdot \left(A_{21}\bar{\mathbf{c}} + A_{22}\hat{\mathbf{c}}\right). \tag{71e}$$

To make the comparison with the equations reported by Öttinger in Ref. [22] (obtained in the nonrelativistic limit of the corresponding relativistic ones), we have to choose the three phenomenological coefficients as follows:

$$A_1 = \frac{1}{T}\cdot\frac{1}{\tau_1},\ \ A_{21} = \frac{1}{T}\cdot\frac{1}{\tau_0},\ \ A_{22} = \frac{1}{T}\cdot\frac{1}{\tau_2}, \tag{72}$$

with τ_0, τ_1, τ_2 denoting the relaxation times driving changes in $\mathbf{w}$, $\bar{\mathbf{c}}$ and $\hat{\mathbf{c}}$, respectively. Then, we see that Eq. (71) contain the extra term $-\mathbf{R}\cdot\mathbf{j}$ in the dynamic equation for $\mathbf{w}$ and the corresponding term $+\frac{1}{T}\mathbf{R}\colon\mathbf{jj}$ in the entropy equation. The set of Eq. (71) without the terms involving the $\mathbf{R}$ tensor has been employed by Öttinger [22] to study the hydrodynamics of a fluid whose internal energy density u, see Eq. (62), is described by an equation of the form $u = u_0 + \frac{1}{2}\alpha^{(1)}I_4 + \frac{1}{2}\alpha^{(2)}I_2$, implying that $\sigma_2 = 2\alpha^{(2)}$ and $\sigma_4 = 2\alpha^{(1)}$. For such a fluid, the thermal conductivity is identified as $\lambda = \sigma_4 T\tau_1$, the viscosity as $\lambda = 2\sigma_2\tau_2$, and the dilatational viscosity as $\kappa = \frac{4}{3}\sigma_2\tau_0$.

3.2 Relativistic formulation

3.2.1 Definitions of fluid properties and some important thermodynamic considerations

To include viscosity, we take the vector of state variables to be again $\mathbf{x} = \{\rho, \mathbf{M}, s, \mathbf{w}, \mathbf{c}\}$ but now we will elevate the covariant vector $\mathbf{w}$ to a four-vector and the mechanical tensor $\mathbf{c}$ to a four-by-four tensor. The time-like component of $\mathbf{w}$ will be exactly the same as before. For the mechanical tensor $c_{\mu\nu}$, on the other hand, which will be closely related to the velocity gradient tensor, we will assume the following form:

$$\left(c_{\mu\nu}\right) = \begin{pmatrix} \frac{\mathbf{v}}{c}\cdot(\mathbf{c}-\mathbf{1})\cdot\frac{\mathbf{v}}{c} & (\mathbf{1}-\mathbf{c})\cdot\frac{\mathbf{v}}{c} \\ (\mathbf{1}-\mathbf{c})\cdot\frac{\mathbf{v}}{c} & \mathbf{c} \end{pmatrix}. \tag{73}$$

This has been constructed such that its time-like components c_{0j} and c_{j0} obey dynamic evolution equations whose structure is very similar to that of the space components c_{ij}. Moreover, the c_{00} element was selected such that $c_{\mu\nu}$ satisfies the following property:

$$c_{\mu\nu}u^{\nu} = u_{\mu}. \tag{74}$$

Of key importance for the subsequent analysis is the definition of the energy-momentum tensor. To this, we assume again expression [21] but with the understanding now that $P^{\mu\nu}$ must be defined differently in order to account for the presence of the new variable $c_{\mu\nu}$. We also assume that the thermodynamic state of the fluid in the local rest frame is described by a general relationship for its internal energy u_f as a function of the independent variables of the form:

$$u_f = u_f\left(\rho_f, s_f, I_1, I_2, I_3, I_4, I_5, I_6\right), \tag{75}$$

where the six invariant quantities $I_1, I_2, I_3, I_4, I_5, I_6$ are defined exactly as in the nonrelativistic case:

$$I_1 = \mathrm{tr}\left(\mathbf{c}_f\right),\ \ I_2 = \mathrm{tr}\left(\mathbf{c}_f^2\right),\ \ I_3 = \mathrm{tr}\left(\mathbf{c}_f^3\right),\ \ I_4 = \mathbf{w}_f^2,\ \ I_5 = \mathbf{w}_f\cdot\mathbf{c}_f\cdot\mathbf{w}_f,$$
$$I_6 = \mathbf{w}_f\cdot\mathbf{c}_f^2\cdot\mathbf{w}_f, \tag{76}$$

with the subscript f used to emphasize that the corresponding fluid properties are evaluated in the rest frame. How the scalars appearing in Eq. (75) can be evaluated in any reference frame is discussed by Öttinger [22]. We also define the corresponding conjugate variables in the comoving frame, exactly as we did in the nonrelativistic case:

$$\mu_f = \frac{\partial u_f}{\partial \rho_f},\ \ T_f = \frac{\partial u_f}{\partial s_f},\ \ \sigma_{f,1} = 2\frac{\partial u_f}{\partial I_1},\ \ \sigma_{f,2} = 4\frac{\partial u_f}{\partial I_2},\ \ \sigma_{f,3} = 6\frac{\partial u_f}{\partial I_3},$$
$$\sigma_{f,4} = 4\frac{\partial u_f}{\partial I_4},\ \ \sigma_{f,5} = 2\frac{\partial u_f}{\partial I_5},\ \ \sigma_{f,6} = 2\frac{\partial u_f}{\partial I_6}. \tag{77}$$

Moreover, we demand the same Euler equation as the one expressed by Eq. (25).

Then, motivated by Eq. (68), we assume the following generalization of the stress tensor:

$$\begin{aligned} P_{\mu\nu} = {} & p_f \eta_{\mu\nu} + (c_{\mu\lambda} - \eta_{\mu\lambda})(\sigma_{f,1} c^0 + \sigma_{f,2} c^1 + \sigma_{f,3} c^2)^{\lambda}_{\nu} + (\sigma_{f,4} - \sigma_{f,5}) w_\mu w_\nu \\ & + (\sigma_{f,5} - \sigma_{f,6})(w_\mu w^\lambda c_{\lambda\nu} + c_{\mu\lambda} w^\lambda w_\nu) \\ & + \sigma_{f,6}\Big(w_\mu w^\lambda c^2_{\lambda\nu} + c^2_{\mu\lambda} w^\lambda w_\nu + c_{\mu\alpha} w^\alpha w^\beta c_{\beta\nu} \Big). \end{aligned} \tag{78}$$

This, in turn, defines the total energy of the relativistic fluid which is given again by Eq. (31), with the energy-momentum tensor described by Eq. (21). Moreover, the momentum vector components are

$$M_i = (T^{00} - P^{00}) \frac{v_i}{c^2} + \frac{1}{c} P^{0i}. \tag{79}$$

The next step is to compute the functional derivatives $\frac{\delta H}{\delta \mathbf{x}}$ that are needed in the construction of the Poisson and dissipative brackets. These have already been calculated by Öttinger [22] and read

$$\frac{\delta H}{\delta \rho} = \frac{c^2 + \mu_f}{\gamma}, \tag{80a}$$

$$\frac{\delta H}{\delta \mathbf{M}} = \mathbf{v}, \tag{80b}$$

$$\frac{\delta H}{\delta s} = \frac{T_f}{\gamma}, \tag{80c}$$

$$\frac{\delta H}{\delta \mathbf{w}} = \mathbf{j}, \tag{80d}$$

$$\frac{\delta H}{\delta \mathbf{c}} = \frac{1}{2} \boldsymbol{\tau}, \tag{80e}$$

where the heat flux vector is given now as

$$j_i = \Big(\sigma^{i\mu} - \frac{v_i}{c} \sigma^{0\mu} \Big) w_\mu, \tag{81}$$

where

$$\sigma^{\mu\nu} = (\sigma_{f,1} c^0 + \sigma_{f,2} c^1 + \sigma_{f,3} c^2)^{\mu\nu}. \tag{82}$$

Also,

$$\tau_{ij} = \phi^{ij} - \phi^{i0} \frac{v_j}{c} - \frac{v_i}{c} \phi^{0j} + \phi^{00} \frac{v_i}{c} \frac{v_j}{c}, \tag{83}$$

with

$$\phi^{\mu\nu} = (\sigma_{f,1} c^0 + \sigma_{f,2} c^1 + \sigma_{f,3} c^2)^{\mu\nu} + \sigma_{f,5} w^\mu w^\nu + \sigma_{f,6} (w^\mu w_\lambda c^{\lambda\nu} + c^{\mu\lambda} w_\lambda w^\nu). \tag{84}$$

The space components ϕ^{ij} of the four-vector tensor $\phi^{\mu\nu}$ do not coincide with the stress tensor components τ^{ij} defined in the nonrelativistic case because they have been corrected to account for the fact that in the definition of Eq. (73) for $c_{\mu\nu}$, the time-like components c_{0j} and c_{0j} are obtained from the product $(\mathbf{1} - \mathbf{c}) \cdot \frac{\mathbf{v}}{c}$. Moreover, if we define [22]

$$\tilde{\sigma}_{ij} = \sigma^{ij} - \sigma^{i0} \frac{v_j}{c} - \frac{v_i}{c} \sigma^{0j} + \sigma^{00} \frac{v_i}{c} \frac{v_j}{c}, \tag{85}$$

then, we recover the following analogue of Eq. (40a):

$$\mathbf{j} = \tilde{\boldsymbol{\sigma}} \cdot \Big(\mathbf{w} - \gamma T_f \frac{\mathbf{v}}{c^2} \Big). \tag{86}$$

Consequently, the tensor $\tilde{\boldsymbol{\sigma}}$ in the case of a viscous fluid is the generalization of the tensor $\sigma_f \Big(\mathbf{1} - \frac{\mathbf{v}\mathbf{v}}{c^2} \Big)$ in the case of an inviscid fluid.

Now, we are in a position to fix the time-like component of the four-vector w^μ as well as the fluid entropy in the comoving frame. To this, we demand the same generalized (relativistic hydrodynamic-thermodynamic)

Euler expression at the level of the total energy of the relativistic fluid as before, see Eq. (35). In Appendix B we examine under what conditions such an equation can apply, and we find that Eq. (35) is identically satisfied if we chose:

$$s = \gamma s_f + \frac{J^0}{c}(\equiv S^0), \tag{87}$$

where

$$J^\mu = -\frac{c}{T_f}\left(P^{\mu\nu}u_\nu + u_\mu u_\alpha P^{\alpha\beta}u_\beta\right). \tag{88}$$

On the other hand, no extra constraints appear for the time-like component of w^μ, thus, we take this to satisfy the same equation as before, Eq. (36). Equations (87) and (88) were the missing building blocks before applying the generalized bracket formalism to derive the relativistic hydrodynamic equations. In fact, Eq. (87) suggests the following expression for the four-vector entropy of the relativistic fluid:

$$S^\mu = s_f u^\mu + \frac{J^\mu}{c}. \tag{89}$$

It is also a rather straightforward exercise to show that [22]:

$$J^\mu = \sigma^{\mu\nu}w_\nu + u^\mu u_\alpha \sigma^{\alpha\beta}w_\beta. \tag{90}$$

3.2.2 Poisson bracket and the corresponding reversible part of the transport equations

We are now in a position to write down the transport equations in the comoving frame. By assuming the same form of the Poisson bracket as in Eq. (59) and by making use of Eq. (42), the reversible part of the dynamic equations reads

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}\rho), \tag{91a}$$

$$\frac{\partial \mathbf{M}}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}\,\mathbf{M}) - \frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{P}, \tag{91b}$$

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot\left(\mathbf{v}\,s + \mathbf{j}\right), \tag{91c}$$

$$\frac{\partial \mathbf{w}}{\partial t} = -\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}\mathbf{w} - \boldsymbol{\kappa}^T\cdot\mathbf{w} - \frac{\partial}{\partial \mathbf{r}}\left(\frac{T_f}{\gamma}\right), \tag{91d}$$

$$\frac{\partial \mathbf{c}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{c} + \mathbf{c}\cdot\boldsymbol{\kappa} + \boldsymbol{\kappa}^T\cdot\mathbf{c} = \boldsymbol{\kappa} + \boldsymbol{\kappa}^T, \tag{91e}$$

where $\mathbf{P}$ is the stress tensor defined by an equation very similar to Eq. (68) above, namely,

$$\begin{aligned}\mathbf{P} = {} & p_f\mathbf{1} + \left(\sigma_{f,1}\mathbf{1} + \sigma_{f,2}\mathbf{c} + \sigma_{f,3}\mathbf{c}^2\right)\cdot(\mathbf{c}-\mathbf{1}) \\ & + \left(\sigma_{f,4} - \sigma_{f,5}\right)\mathbf{ww} + \left(\sigma_{f,5} - \sigma_{f,6}\right)(\mathbf{ww}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{ww}) \\ & + \sigma_{f,6}\left(\mathbf{ww}\cdot\mathbf{c}^2 + \mathbf{c}^2\cdot\mathbf{ww} + \mathbf{c}\cdot\mathbf{ww}\cdot\mathbf{c}\right),\end{aligned} \tag{92}$$

whose components match the space components of the stress tensor defined by Eq. (78). The explicit Lorentz-covariant form of the above transport equations is

$$\partial_\mu\left(\rho_f u^\mu\right) = 0, \tag{93a}$$

$$\partial_\mu\left[\left(\rho_f c^2 + u_f - u_\alpha P^{\alpha\beta}u_\beta\right)u^\mu u^\nu + P^{\mu\nu}\right] = 0, \tag{93b}$$

$$\partial_\mu\left(s_f u^\mu + \frac{J^\mu}{c}\right) = 0, \tag{93c}$$

$$u^\mu\left(\partial_\nu w_\mu - \partial_\mu w_\nu\right) = 0, \tag{93d}$$

$$u^\lambda\left(\partial_\lambda c_{\mu\nu} - \partial_\mu c_{\lambda\nu} - \partial_\nu c_{\mu\lambda}\right) = 0. \tag{93e}$$

3.2.3 Dissipative bracket and full form of transport equations

To complete the above set of equations, we need to add the dissipative terms. Given that we make use of the same set of structural variables as in the nonrelativistic case, the dissipative bracket is taken to be the same as the one defined in Section 3.1.3, see Eq. (70), without the term proportional to A_1, and with the understanding that (due to the Onsager–Casimir reciprocity principle) the tensor $\mathbf{R}$ should be again symmetric, see Eq. (11c):

$$\begin{aligned}[F,G] = &-\int R_{ij}\frac{\delta F}{\delta w_i}\frac{\delta G}{\delta w_j}\mathrm{d}V + \int \frac{1}{T}\frac{\delta F}{\delta s}R_{ij}\frac{\delta G}{\delta w_i}\frac{\delta G}{\delta w_j}\mathrm{d}V \\ &-\int \left(A_{21}\left(\bar{c}_{ij}-\delta_{ij}\right)+A_{22}\left(\hat{c}_{ij}-\delta_{ij}\right)\right)\frac{\delta F}{\delta c_{ij}}\frac{\delta G}{\delta s}\mathrm{d}V \\ &+\int \frac{1}{T}\frac{\delta F}{\delta s}\left(A_{21}\left(\bar{c}_{ij}-\delta_{ij}\right)+A_{22}\left(\hat{c}_{ij}-\delta_{ij}\right)\right)\frac{\delta G}{\delta c_{ij}}\frac{\delta G}{\delta s}\mathrm{d}V\cdot \end{aligned} \tag{94}$$

From this dissipation bracket, the resulting dynamic equations read

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}\rho), \tag{95a}$$

$$\frac{\partial \mathbf{M}}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}\,\mathbf{M}) - \frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{P}, \tag{95b}$$

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\cdot\left(\mathbf{v}\,s+\mathbf{j}\right) + \frac{1}{T}\mathbf{R}{:}\,\mathbf{jj} + +\left(A_{21}\bar{\mathbf{c}}+A_{22}\hat{\mathbf{c}}\right){:}\,\boldsymbol{\tau}, \tag{95c}$$

$$\frac{\partial \mathbf{w}}{\partial t} = -\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}\mathbf{w} - \boldsymbol{\kappa}^T\cdot\mathbf{w} - \frac{\partial}{\partial \mathbf{r}}\left(\frac{T_f}{\gamma}\right) - \mathbf{R}\cdot\mathbf{j}, \tag{95d}$$

$$\frac{\partial \mathbf{c}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{c} + \mathbf{c}\cdot\boldsymbol{\kappa} + \boldsymbol{\kappa}^T\cdot\mathbf{c} = \boldsymbol{\kappa} + \boldsymbol{\kappa}^T - T\cdot\left(A_{21}\bar{\mathbf{c}}+A_{22}\hat{\mathbf{c}}\right). \tag{95e}$$

The final step is to determine the matrix $\mathbf{R}$ so that a covariant set of transport equations arises. One way to achieve this is to take advantage of the decomposition implied by Eq. (86) for the vector $\mathbf{j}$ and take the matrix $\mathbf{R}$ to be the inverse of the $\tilde{\boldsymbol{\sigma}}$, namely,

$$\mathbf{R} \equiv A_0\cdot\tilde{\boldsymbol{\sigma}}^{-1}, \tag{96}$$

where A_0 a constant. Then, the entropy production term due to vector $\mathbf{w}$ becomes proportional to $\left(w_i - \gamma T_f\frac{v_i}{c^2}\right)j_i$, which is equal to

$$\left(w_i - \gamma T_f\frac{v_i}{c^2}\right)j_i = \left(w_i - \gamma T_f\frac{v_i}{c^2}\right)\left(J_i - u^i u_\alpha \sigma^{\alpha\beta}w_\beta - \frac{v_i}{c}\sigma^{0\mu}w_\mu\right) = w_\mu J^\mu, \tag{97}$$

and thus Lorentz-covariant. Moreover, we can introduce the traceless and isotropic parts of $c_{\mu\nu}$ through

$$\bar{c}_{\mu\nu} = \frac{1}{3}\left(c^\lambda_\lambda - 1\right)\left(\eta_{\mu\nu} + u_\mu u_\nu\right) \tag{98a}$$

and

$$\hat{c}_{\mu\nu} = c_{\mu\nu} + u_\mu u_\nu - \bar{c}_{\mu\nu}, \tag{98b}$$

respectively. Then, for the spatial components we have

$$\bar{\mathbf{c}} = \frac{1}{3}\mathrm{tr}(\mathbf{c})\left(\mathbf{1} + \gamma^2\frac{\mathbf{vv}}{c^2}\right), \tag{99a}$$

and

$$\hat{\mathbf{c}} = \mathbf{c} + \gamma^2\frac{\mathbf{vv}}{c^2} - \bar{\mathbf{c}}. \tag{99b}$$

With these definitions, one can prove that [22]

$$\bar{c}_{ij}\tau_{ij} = \bar{c}_{\mu\nu}\phi^{\mu\nu}, \tag{100a}$$

$$\hat{c}_{ij}\tau_{ij} = \hat{c}_{\mu\nu}\phi^{\mu\nu}, \tag{100b}$$

as well as that

$$\left(c_{ij} + \gamma^2 \frac{v_i v_j}{c^2}\right)\tau_{ij} = c_{\mu\nu}\phi^{\mu\nu}. \tag{100c}$$

A direct proof of Eq. (100c) is provided in Appendix C. Equations (97), (100a), and (100b) prove that both entropy production terms on the right-hand side of Eq. (95c) are covariant. Then, the five dynamic equations become

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v}\rho), \tag{101a}$$

$$\frac{\partial \mathbf{M}}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v}\,\mathbf{M}) - \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{P}, \tag{101b}$$

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v}\,s + \mathbf{j}) + \frac{1}{T}A_0\left(\mathbf{w} - \gamma T_f^2 \frac{\mathbf{v}}{c^2}\right) \cdot \mathbf{j} + \left(A_{21}\bar{\mathbf{c}} + A_{22}\hat{\mathbf{c}}\right) : \boldsymbol{\tau}, \tag{101c}$$

$$\frac{\partial \mathbf{w}}{\partial t} = -\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}}\mathbf{w} - \boldsymbol{\kappa}^T \cdot \mathbf{w} - \frac{\partial}{\partial \mathbf{r}}\left(\frac{T_f}{\gamma}\right) - A_0\left(\mathbf{w} - \gamma T_f \frac{\mathbf{v}}{c^2}\right), \tag{101d}$$

$$\frac{\partial \mathbf{c}}{\partial t} + \mathbf{v} \cdot \nabla\mathbf{c} + \mathbf{c} \cdot \boldsymbol{\kappa} + \boldsymbol{\kappa}^T \cdot \mathbf{c} = \boldsymbol{\kappa} + \boldsymbol{\kappa}^T - T \cdot \left(A_{21}\bar{\mathbf{c}} + A_{22}\hat{\mathbf{c}}\right), \tag{101e}$$

or, in Lorentz-covariant form,

$$\partial_\mu\left(\rho_f u^\mu\right) = 0, \tag{102a}$$

$$\partial_\mu\left[\left(\rho_f c^2 + u_f - u_\alpha P^{\alpha\beta} u_\beta\right)u^\mu u^\nu + P^{\mu\nu}\right] = 0, \tag{102b}$$

$$\partial_\mu\left(s_f u^\mu + \frac{J^\mu}{c}\right) = A_0 w_\mu J^\mu + A_{21}\bar{c}_{\mu\nu}\phi^{\mu\nu} + A_{22}\hat{c}_{\mu\nu}\phi^{\mu\nu}, \tag{102c}$$

$$u^\mu\left(\partial_\nu w_\mu - \partial_\nu w_\mu\right) = -A_2\left(\eta_{\nu\lambda} + u_\nu u_\lambda\right)w^\lambda, \tag{102d}$$

$$u^\lambda\left(\partial_\lambda c_{\mu\nu} - \partial_\mu c_{\lambda\nu} - \partial_\nu c_{\mu\lambda}\right) = -\frac{A_{21}T}{c}\bar{c}_{\mu\nu} - \frac{A_{22}T}{c}\hat{c}_{\mu\nu}. \tag{102e}$$

The set of relativistic equations, Eq. (102), is exactly the same as that derived by Öttinger from GENERIC [21]. However, we have to keep in mind that these were obtained by choosing the phenomenological friction matrix **R** to satisfy Eq. (96), implying that one could in principle think of other choices of **R** that could render the transport equations for the entropy and the thermal vector covariant, and this is something worth pursuing further. We also observe that in the limit of infinite speed light $\left(\frac{v}{c} \to 0\right)$, the nonrelativistic analogue of Eq. (102) reduces to that of Öttinger, which is quite pleasing.

4 Discussion

It is remarkable that one can use exactly the same form of the Poisson and dissipative brackets in order to formulate time-evolution equations in the two cases (relativistic and nonrelativistic). Also remarkable is the fact that one can use the generalized Euler equation, Eq. (35), to get guidance as to how to define the time-like component of the thermal vector and the entropy four-vector. On the other hand, the full consistency of the final relativistic equations between the two formalisms of nonequilibrium thermodynamics reveals once more their close connection and equivalence, despite some striking differences. This also implies that similar conclusions can be drawn from the present work as those pointed out by Öttinger from his original work on the structural compatibility of GENERIC with special relativity [21]. That is, that even in the absence of viscous effects, the classical theory of Eckart [23], the second-order theory of Israel [24], and the equations of extended irreversible thermodynamics [25] and of kinetic theory [26] do not possess the full nonequilibrium structure of the equations derived from the generalized bracket and the GENERIC formalisms, although both Israel's theory and extended irreversible thermodynamics are very similar in structure while kinetic theory can provide the linearized form of the equations.

5 Conclusions

Using as an example the case of an imperfect viscous fluid with heat flow, we have shown that the generalized bracket formalism of nonequilibrium thermodynamics is fully compatible with special relativity. Indeed, by appropriately choosing the stress tensor and the entropy-current four-vector we have been able to formulate a set of Lorentz-covariant equations for the three hydrodynamic fields (density, momentum and entropy) and the two generalized force variables (the thermal vector and the mechanical tensor) that are consistent with the fundamental evolution equation of the formalism. In our work, the Hamiltonian (i.e., the single generator of the formalism) has been represented by the time-component of the energy-momentum tensor while the internal energy density has allowed us to define several auxiliary variables playing the role of relativistic field properties.

In the more general context of the formalism, it appears that to define Lorentz-covariant equations, one can start with the same Poisson bracket and the same dissipation bracket as for the corresponding nonrelativistic system. However, for the final transport equations to satisfy Lorentz covariance, the dissipation rates in the dissipation matrix must be restricted to specific forms. In fact, in the relativistic case the matrix describing resistance to heat flow is inherently anisotropic as it must depend explicitly on the velocity field in order for the final equations to be Lorentz-covariant. It also appears that the form of the four-vector entropy current can be correctly guessed by invoking a generalized hydrodynamic-thermodynamic Euler equation, which can tremendously help identify suitable forms of the relevant rates.

In the future, we plan to carry out a systematic stability analysis of the final time-evolution equations to define the range of physically meaningful model parameters for which stable solutions are computed not only around the zero-velocity field but also when boosts are considered.

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Appendix A

To find under what conditions Eq. (35) holds, we start from its left-hand side (LHS) which with the help of Eqs. ((33a)–(33c)) becomes

$$\begin{aligned}\text{LHS} &= \rho\frac{c^2+\mu_f}{\gamma} + \mathbf{M}\cdot\mathbf{v} + s\frac{T_f}{\gamma}\\ &= \rho_f c^2 + \rho_f\mu_f + \left(\left(T^{00} + p_f - \sigma_f w_0^2\right)\frac{\mathbf{v}}{c^2} - \sigma_f w_0\frac{\mathbf{w}}{c}\right)\cdot\mathbf{v} + s\frac{T_f}{\gamma}\\ &= \rho_f c^2 + \rho_f\mu_f + \left(T^{00} + p_f - \sigma_f w_0^2\right)\frac{v^2}{c^2} - \sigma_f w_0\frac{\mathbf{w}\cdot\mathbf{v}}{c} + s\frac{T_f}{\gamma}.\end{aligned} \tag{A.1}$$

But from Eqs. (22) and (26) we find that

$$T^{00} = \left(\rho_f c^2 + u_f + p_f + \sigma_f\left(u_\alpha w^\alpha\right)^2\right)\gamma^2 - p_f + \sigma_f w_0^2, \tag{A.2}$$

or, using Eq. (25),

$$T^{00}=\left(\rho_f c^2+T_f s_f+\mu_f\rho_f+\sigma_f\left(u_\alpha w^\alpha\right)^2\right)\gamma^2-p_f+\sigma_f w_0^2. \tag{A.3}$$

Thus,

$$\rho_f c^2+\rho_f\mu_f=\frac{T^{00}+p_f-\sigma_f w_0^2}{\gamma^2}-s_f T_f-\sigma_f\left(u_\alpha w^\alpha\right)^2. \tag{A.4}$$

Substituting this back into Eq. (A.1) yields

$$\begin{aligned}\text{LHS}=&\frac{T^{00}+p_f-\sigma_f w_0^2}{\gamma^2}-s_f T_f-\sigma_f\left(u_\alpha w^\alpha\right)^2+\left(T^{00}+p_f-\sigma_f w_0^2\right)\frac{v^2}{c^2}\\&-\sigma_f w_0\frac{\mathbf{w}\cdot\mathbf{v}}{c}+s\frac{T_f}{\gamma}\\=&T^{00}+p_f-\sigma_f w_0^2-s_f T_f-\sigma_f\left(u_\alpha w^\alpha\right)^2-\sigma_f w_0\frac{\mathbf{w}\cdot\mathbf{v}}{c}+s\frac{T_f}{\gamma}.\end{aligned} \tag{A.5}$$

Interestingly, if we choose w^0 so that Eq. (36) is satisfied and s according to Eq. (37), the last five terms on the right-hand side (RHS) of Eq. (A.5) cancel out identically, nicely leading to the RHS of Eq. (35).

Appendix B

To find under what conditions Eq. (35) holds for an imperfect viscous fluid with heat conduction, we start from its left-hand side which with the help of Eqs. ((80a)–(80c)) and (79) becomes:

$$\begin{aligned}\text{LHS}&=\rho\frac{c^2+\mu_f}{\gamma}+\mathbf{M}\cdot\mathbf{v}+s\frac{T_f}{\gamma}\\&=\rho_f c^2+\rho_f\mu_f+\left(\left(T^{00}-P^{00}\right)\frac{v_i}{c^2}+\frac{1}{c}P^{0i}\right)v_i+s\frac{T_f}{\gamma}\\&=\rho_f c^2+\rho_f\mu_f+\left(T^{00}-P^{00}\right)\frac{v^2}{c^2}+\frac{1}{c}P^{0i}v_i+s\frac{T_f}{\gamma}.\end{aligned} \tag{B.1}$$

But from Eq. (21), we find that:

$$T^{00}=\left(\rho_f c^2+u_f-u_\alpha P^{\alpha\beta}u_\beta\right)\gamma^2+P^{00}, \tag{B.2}$$

or, using Eq. (25),

$$T^{00}=\left(\rho_f c^2+T_f s_f-p_f+\rho_f\mu_f-u_\alpha P^{\alpha\beta}u_\beta\right)\gamma^2+P^{00}. \tag{B.3}$$

Thus,

$$\rho_f c^2+\rho_f\mu_f=\frac{T^{00}-P^{00}}{\gamma^2}-s_f T_f+p_f+u_\alpha P^{\alpha\beta}u_\beta. \tag{B.4}$$

Substituting this back into Eq. (B.1) yields

$$\begin{aligned}\text{LHS}&=\frac{T^{00}-P^{00}}{\gamma^2}-s_f T_f+p_f+u_\alpha P^{\alpha\beta}u_\beta+\left(T^{00}-P^{00}\right)\frac{v^2}{c^2}+\frac{1}{c}P^{0i}v_i+s\frac{T_f}{\gamma}\\&=T^{00}-P^{00}-s_f T_f+p_f+u_\alpha P^{\alpha\beta}u_\beta+\frac{1}{c}P^{0i}v_i+s\frac{T_f}{\gamma}\\&=T^{00}+p_f-\left(P^{00}-u_\alpha P^{\alpha\beta}u_\beta-\frac{1}{c}P^{0i}v_i\right)+\left(s\frac{T_f}{\gamma}-s_f T_f\right).\end{aligned} \tag{B.5}$$

If we define

$$J^0=-\frac{c}{T_f}\left(P^{0\nu}u_\nu+u^0u_\alpha P^{\alpha\beta}u_\beta\right),$$

then,

$$J^0 = -\frac{c}{T_f}\left(-\gamma P^{00} + \gamma\frac{v_i}{c}P^{0i} + \gamma u_\alpha P^{\alpha\beta}u_\beta\right) = \gamma\frac{c}{T_f}\left(P^{00} - \frac{v_i}{c}P^{0i} - u_\alpha P^{\alpha\beta}u_\beta\right).$$

Thus, Eq. (B.5) becomes:

$$\begin{aligned}\text{LHS} &= T^{00} + p_f - \frac{1}{\gamma}\frac{T_f}{c}J^0 + s\frac{T_f}{\gamma} - s_f T_f \\ &= T^{00} + p_f + \frac{T_f}{\gamma}\left(s - \gamma s_f - \frac{J^0}{c}\right),\end{aligned}$$

implying that Eq. (35) holds provided that we choose the entropy as $s \equiv \gamma s_f + \frac{J^0}{c}(= S^0)$, i.e., according to Eq. (87) in the main text.

Appendix C

With the help of Eq. (83), we have:

$$\begin{aligned}(c_{\mu\nu} + u_\mu u_\nu)\phi^{\mu\nu} &= (c_{ij} + u_i u_j)\phi^{ij} + (c_{i0} + u_i u_0)\phi^{i0} + (c_{0j} + u_0 u_j)\phi^{0j} + (c_{00} + u_0 u_0)\phi^{00} \\ &= \left(c_{ij} + \gamma^2\frac{v_i v_j}{c^2}\right)\left(\tau^{ij} + \phi^{i0}\frac{v_j}{c} + \phi^{0j}\frac{v_i}{c} - \phi^{00}\frac{v_i}{c}\frac{v_j}{c}\right) \\ &\quad + \left(c_{i0} - \gamma^2\frac{v_i}{c}\right)\phi^{i0} + \left(c_{0j} - \gamma^2\frac{v_j}{c}\right)\phi^{0j} + (c_{00} + \gamma^2)\phi^{00},\end{aligned} \tag{C.1}$$

thus, to prove Eq. (100c), we must show that

$$\begin{aligned}&\left(c_{ij} + \gamma^2\frac{v_i v_j}{c^2}\right)\left(\phi^{i0}\frac{v_j}{c} + \phi^{0j}\frac{v_i}{c} - \phi^{00}\frac{v_i}{c}\frac{v_j}{c}\right) \\ &\quad + \left(c_{i0} - \gamma^2\frac{v_i}{c}\right)\phi^{i0} + \left(c_{0j} - \gamma^2\frac{v_j}{c}\right)\phi^{0j} + (c_{00} + \gamma^2)\phi^{00} = 0.\end{aligned} \tag{C.2}$$

If we carry out the calculations and collect terms, the LHS of Eq. (C.2), we find:

$$\begin{aligned}\text{LHS} &= \phi^{i0}\left(c_{ij}\frac{v_j}{c} + \gamma^2\frac{v_i}{c}\frac{v_j v_j}{c^2} + c_{i0} - \gamma^2\frac{v_i}{c}\right) + \phi^{0j}\left(c_{ij}\frac{v_i}{c} + \gamma^2\frac{v_j}{c}\frac{v_i v_i}{c^2} + c_{0j} - \gamma^2\frac{v_j}{c}\right) \\ &\quad + \phi^{00}\left(-c_{ij}\frac{v_i v_j}{c^2} - \gamma^2\left(\frac{v_i}{c}\right)^2\left(\frac{v_j}{c}\right)^2 + c_{00} + \gamma^2\right),\end{aligned} \tag{C.3}$$

or, equivalently,

$$\begin{aligned}\text{LHS} &= \phi^{i0}\left(c_{ij}\frac{v_j}{c} + \gamma^2\frac{v_i}{c}\frac{v^2}{c^2} + c_{i0} - \gamma^2\frac{v_i}{c}\right) + \phi^{0j}\left(c_{ij}\frac{v_i}{c} + \gamma^2\frac{v_j}{c}\frac{v^2}{c^2} + c_{0j} - \gamma^2\frac{v_j}{c}\right) \\ &\quad + \phi^{00}\left(-c_{ij}\frac{v_i v_j}{c^2} - \gamma^2\frac{v^2}{c^2}\frac{v^2}{c^2} + c_{00} + \gamma^2\right).\end{aligned} \tag{C.4}$$

But, by definition of the tensor $c_{\mu\nu}$,

$$\begin{aligned}c_{ij}\frac{v_j}{c} + c_{i0} &= \frac{v_i}{c} \\ c_{ij}\frac{v_i}{c} + c_{0j} &= \frac{v_j}{c} \\ c_{00} &= \frac{v_i c_{ij} v_j}{c^2} - \frac{v^2}{c^2} - 1.\end{aligned} \tag{C.5}$$

Thus, Eq. (C.4) becomes

$$\mathrm{LHS} = \phi^{i0}\frac{v_i}{c}\left(1+\gamma^2\frac{v^2}{c^2}-\gamma^2\right) + \phi^{0j}\frac{v_j}{c}\left(1+\gamma^2\frac{v^2}{c^2}-\gamma^2\right) + \phi^{00}\left(-\gamma^2\frac{v^2}{c^2}\frac{v^2}{c^2}-\frac{v^2}{c^2}-1+\gamma^2\right). \tag{C.6}$$

And if we finally substitute Eq. (18) in the main text for γ in each of the three parentheses on the RHS of Eq. (C.6), we find that all of them are zero. Thus, $\mathrm{LHS} = 0$, which is what we wanted to prove.

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